# Balanced growth for solutions of nonautonomous partial differential equations

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**Abstract**

We show balanced growth for solutions of some nonautonomous partial differential equations which in certain cases also describe the dynamics of structured populations.
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## 1. Introduction

In the present note we investigate the asymptotic behaviour of solutions of certain linear initial boundary value problems (IBVP for short). In particular, we are interested in when solutions blow up in time but after a normalization they converge to a final distribution. In case of autonomous IBVP-s this phenomenon, which is quite important both from the theoretical ([1],[2],[3]) and application point of view ([4],[5]), is usually described in the framework of linear semigroup theory ([6],[7]). Here we present a different approach which works in certain cases of nonautonomous problems which cannot be covered in the framework of semigroup theory.

To be more specific we are interested in the asymptotic behaviour of solutions of the following nonautonomous IBVP.

$$\begin{cases} a(t)u_t(x,t) + b(x)u_x(x,t) + c(x)u(x,t) = 0, & t > 0,\ x > 0, \\ u(0,t) = B(t), & t > 0, \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{1.1}$$

where $a, b, c > 0$ are smooth enough for the upcoming analysis and for (1.1) to be well-posed in $L^1$, as well. We assume that $B > 0$ on which we will make some additional conditions later on.

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**Definition 1.1** *The solution* $u$ *of* (1.1) *exhibits balanced growth if the following limit exists for every* $0 \leq x_1 < x_2 < \infty$

$$\lim_{t \to \infty} \frac{\int_{x_1}^{x_2} u(x,t)dx}{\int_0^\infty u(x,t)dx}, \tag{1.2}$$

*and it is independent of the initial condition* $u_0$ *in* (1.1).

Our definition here is motivated by the applications, namely when (1.1) describes the dynamics of a structured population (see [8],[9],[10],[11] and the references therein for developments in structured population dynamics). In particular, when

$$a(t) \equiv 1,\ b(x) \equiv 1,\ u(0,t) = B(t) = \int_0^\infty \beta(x) u(x,t) dx, \tag{1.3}$$

(1.1) becomes the classical Lotka-Kermack-McKendrick linear age-structured model where $c(x)$ and $\beta(x)$ denote the mortality and fertility of individuals of age $x$, respectively. In this case $u(.,t)$ is the age-distribution of individuals, and balanced growth characterizes the situation when the population blows up in time but the proportion of individuals within any age range, compared to the total population, tends to a limit which just depends on the choosen range but not on the initial condition.

We arrive at the classical definition of balanced exponential growth if in addition to Def.1 we assume that $u$ grows exponentially in time.

Several weaker definitions of balanced exponential growth can be found in [1, 12], in which papers the author characterizes strong and uniform balanced exponential growth using quite deep functional analytic methods.

In most of the literature (see [1],[3],[13] and the references therein) balanced exponential growth (or asynchronous exponential growth) is characterised in the framework of linear semigroup theory. In the framework of semigroup theory the $C_0$ (strongly continuous) semigroup $\{T(t)\}_{t\geq 0}$ on the Banach space $\mathcal{X}$ exhibits balanced exponential growth if there exist $0 < r \in \mathbb{R}$ (often called the Malthusian parameter) and a projection $\Pi$ of $\mathcal{X}$ such that

$$\lim_{t \to \infty} e^{-rt} T(t)x = \Pi x \quad \text{for all} \quad x \in \mathcal{X}. \tag{1.4}$$

In other words, the system which is governed by the semigroup $\{T(t)\}_{t\geq 0}$ admits a global attractor, namely $\Pi(\mathcal{X})$. If, in addition, the operator $\Pi$ has finite rank then $\{T(t)\}_{t\geq 0}$ is said to have asynchronous exponential growth ([3]), AEG for short.

In fact, the existence of such $r > 0$ in (1.4) is related to the existence of a strictly dominant eigenvalue in the spectrum of the generator $A$ of the semigroup. Moreover, if $\{T(t)\}_{t\geq 0}$ is a positive $C_0$ semigroup on the Banach space $\mathcal{X}$ then the spectral bound $s(A)$ of the generator $A$ of $\{T(t)\}_{t\geq 0}$ is real, strictly dominant and it belongs to the spectrum. In other words $s(A)$ is a pole of the resolvent $R(\lambda, A)$. The order of the pole might in general be greater than one.

Recently, in [13] the authors proved using semigroup methods AEG for solutions of a linear age-structured model ($a(t) \equiv 1$, $b(x) \equiv 1$) with delayed birth rate, i.e. when

$$u(0,t) = B(t) = \int_0^\infty \int_{-\tau}^0 \beta(x,\sigma) u(x, t+\sigma) d\sigma dx. \tag{1.5}$$

## 2. Balanced growth for model (1.1)

In [14] we were able to show using quite elementary methods AEG for the linear age-structured population model with the delayed birth rate (1.5) in certain cases. Here we extend this approach for the nonautonomous model (1.1).

**Theorem 2.1** *The solution $u$ of (1.1) exhibits balanced growth if*

$$B(t) = \kappa e^{rt} + f(t), \quad \textit{where} \lim_{t\to\infty} \frac{f(t)}{e^{rt}} = 0, \quad \kappa, r \geq 0 \tag{2.1}$$

*and*

$$\lim_{t\to\infty} \frac{a(t)}{t^\alpha} = 0, \ \forall\ \alpha > 1. \tag{2.2}$$

**Proof.** The ODE system of characteristics of the linear partial differential equation $(1.1)_1$ is

$$\frac{dt}{d\tau} = a(t), \quad \frac{dx}{d\tau} = b(x), \quad \frac{du}{d\tau} + c(x)u = 0, \tag{2.3}$$

which by setting

$$\Gamma_1(t) = \int \frac{1}{a(t)} dt, \quad \Gamma_2(x) = \int \frac{1}{b(x)} dx, \tag{2.4}$$

can be solved as follows

$$\begin{aligned} &t(\tau) = \Gamma_1^{-1}(\tau + t_0), \quad x(\tau) = \Gamma_2^{-1}(\tau + x_0), \\ &u(x(\tau), t(\tau)) = u(x(0), t(0)) \exp\left\{ -\int_0^\tau c(\Gamma_2^{-1}(r)) dr \right\}. \end{aligned} \tag{2.5}$$

Now if we choose the initial conditions $x(0) = x_0 = 0$, $t(0) = t_0$ we arrive at

$$u(x,t) = u(0, \Gamma_1(t) - \Gamma_2(x)) \exp\left\{ -\int_0^{\Gamma_2(x)} c(\Gamma_2^{-1}(r)) dr \right\} = B(\Gamma_1(t) - \Gamma_2(x))\pi(x), \tag{2.6}$$

where

$$\pi(x) = \exp\left\{ -\int_0^{\Gamma_2(x)} c(\Gamma_2^{-1}(r)) dr \right\} = \exp\left\{ -\int_0^x \frac{c(s)}{b(s)} ds \right\}. \tag{2.7}$$

Thus we find that (2.6) is a solution of (1.1) which is independent of the initial condition $u_0$ for all $t$ for which $\Gamma_1(t) > \Gamma_2(x)$.

If $a$ satisfies condition (2.2) then it is easy to see that $\Gamma_1(t) \to \infty$ as $t \to \infty$, thus we have the following:

$$\lim_{t\to\infty} \frac{u(x,t)}{e^{r\Gamma_1(t))}} = \kappa e^{-r\Gamma_2(x)}\pi(x), \text{ for every } x \in (0,\infty). \tag{2.8}$$

In particular,

$$\lim_{t\to\infty} \frac{\int_{x_1}^{x_2} u(x,t)dx}{\int_0^\infty u(x,t)dx} = \frac{\int_{x_1}^{x_2} \lim_{t\to\infty} \frac{B(\Gamma_1(t)-\Gamma_2(x))}{B(\Gamma_1(t))}\pi(x)dx}{\int_0^\infty \lim_{t\to\infty} \frac{B(\Gamma_1(t)-\Gamma_2(x))}{B(\Gamma_1(t))}\pi(x)dx}$$

$$= \frac{\int_{x_1}^{x_2} e^{-r\Gamma_2(x)}\pi(x)dx}{\int_0^\infty e^{-r\Gamma_2(x))}\pi(x)dx} = \frac{\int_{x_1}^{x_2} \exp\left\{-\int_0^x \frac{r+c(s)}{b(s)}ds\right\}dx}{\int_0^\infty \exp\left\{-\int_0^x \frac{r+c(s)}{b(s)}ds\right\}dx}. \tag{2.9}$$

□

**Remark 2.2** Notice that condition (2.1) means that $B$ grows at most exponentially in time, while condition (2.2) requires that $a$ grows at most linearly in time.

**Remark 2.3** This result gives a new approach to handle nonautonomous partial differential equations which cannot be treated in the framework of linear semigroup theory. Moreover, our approach covers the cases of autonomous ($a \equiv 1$) age and size-structured population models if the birth rate $B$ grows in an exponential fashion. This is "probably" the case of most of the structured population models (even with delayed birth rate). See for example Th.5.2 in [10] where such a result (i.e. the birth rate grows exponentially) was proven for the classical age-structured model.

**Remark 2.4** Model (1.1) gives rise to the investigation of size-structured models with time dependent mortality and growth rates, since by setting

$$\widetilde{\gamma}(x,t) = \frac{b(x)}{a(t)}, \quad \widetilde{\mu}(x,t) = \frac{c(x)-b'(x)}{a(t)}$$

(1.1) can be written as

$$u_t(x,t) + (\widetilde{\gamma}(x,t)u(x,t))_x + \widetilde{\mu}(x,t)u = 0, \; u(0,t) = B(t), \; u(x,0) = u_0(x),$$

a size-structured population model with time dependent growth rate $\widetilde{\gamma}$ and mortality $\widetilde{\mu}$. In particular, in case of

$$a(t) = t, \; c(x) = m(x) + b'(x), \quad \text{and} \quad u(0,t) = B(t) = \kappa e^{rt},$$

where $\kappa, r, m(.) > 0$, (1.1) is a nonautonomous size-structured model with growth rate $\gamma(x,t) = \frac{b(x)}{t}$, $t > 0$ and mortality $\mu(x,t) = \frac{m(x)}{t}$, $t > 0$, where the influx of zero (or minimal) size individuals given by $B$ doesn't depend on the standing population. By (2.8) we have

$$\lim_{t\to\infty} \frac{u(x,t)}{t^r} = \kappa e^{-r\Gamma_2(x)}\pi(x), \quad x \in (0,\infty).$$

We find this special example quite remarkable since we have an exponential "input" (the density of zero size individuals) but the population grows polynomially. In fact, $a(t) = t$ is the only case when the solution will grow polynomially despite the exponential boundary condition $B$.

**Remark 2.5** In [12] the author introduced a concept similar to balanced growth, namely the concept of conditional asymptotic equality, to characterize the qualitative behaviour of solutions of certain nonautonomous problems. Conditional asymptotic equality of solutions means that either all solutions tend to zero (extinction) or for any pair $u, v$ of non-negative non-trivial solution one has

$$\lim_{t\to\infty} \frac{u(x,t)}{v(x,t)} = 1, \tag{2.10}$$

uniformly in $x$. Notice that balanced growth (according to Def.1) does not imply asymptotic equality of solutions and vica versa. In case of model (1.1) we can show "weak" asymptotic equality of solutions even without assuming condition (2.1) on $B$. This is because (2.6) provides a formula for the solution $u$ of (1.1) for every $x \in (0,\infty)$ which is independent of the initial condition $u_0$ if condition (2.2) holds. This means that for any pair of non-trivial solutions $u, v$ of (1.1) we have $\lim_{t\to\infty} \frac{u(x,t)}{v(x,t)} = 1$ pointwise in $x$.

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