Richard G. Heck Jr. *Reading Frege's Grundgesetze* Oxford: Oxford University Press, 2012, ISBN: 978-0199233700. Pp. xvii + 296.

Philip Ebert

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Richard G. Heck's *Reading Frege's Grundgesetze* is a fantastic addition to the growing research that focuses primarily on Frege's *Grundgesetze der Arithmetik*. In fact, it is a must read for any Frege scholar, or more broadly any philosopher interested in early analytic philosophy, and logicians as well as mathematicians interested in the history of their field.

The aim of the book is to restore *Grundgesetze* to the place Frege had intended, i.e. his *magnum opus*—his greatest intellectual achievement. Over the second-half of the last century most of Frege scholarship focused mainly on his first two books, *Begriffsschrift* (1879) and *Grundlagen der Arithmetik* (1884), and Frege's three famous articles published in 1891-92, while sidelining Frege's more formal work even when it came to Frege's philosophy of mathematics. So for example, the undoubtedly most influential and most eminent Frege scholar, Sir Michael Dummett writes: "Hence, despite some serious uncertainties, we may consider *Grundlagen* as expressing, with fair accuracy, Frege's mature philosophy of arithmetic, not merely a superseded phase of his thinking." (Dummett, 1991, p.9)

One could reasonable say, then, that *Grundgesetze* was often regarded as a mere formal development of the philosophical ideas outlined in *Grundlagen*, with however one glaring error that made Frege's two volumes filled with "strange signs [and] pages of nothing but alien formulae"(Frege, 2013, vol I, p.XI) even less interesting to a modern reader—those strange signs were part of an inconsistent formal system.

Heck's most ambitious aim in this book is to show that an exclusively *Grundlagen*-centred approach to Frege's mature philosophy in general and his philosophy of mathematics in particular is flawed. Heck's book is a testament to the philosophical richness of Frege's philosophy in his *magnum opus* and he shows in an elegant and convincing way that there is much that we can learn *philosophically* from paying careful attention to Frege's presentation of the formal system of *Grundgesetze* and from studying in detail Frege's proofs. I think one cannot but endorse Heck's claim that Frege scholarship will benefit from a 'course correction' and that a more *Grundgesetze*-centred approach (in combination with the articles published around that time) is a most fruitful way to understanding Frege's mature philosophy, in particular his philosophy of mathematics.[1] So, even if one disagrees with the details of Heck's interpretation (to be summarised below), it will be hard to deny that that there is a wealth of philosophical ideas in Frege's *Grundgesetze* worthy of the label Frege's *mature* philosophy of logic and mathematics.

Reading Frege's Grundgesetze is structured in two parts. The first part covers part I of *Grundgesetze* in which Frege introduces his formal system. It contains four chapters and draws on three previously published articles—these were taken apart, rewritten, reassembled, and expanded by discussing work published since their publication. So this part reads much like a book and not like a mere collection of essays. The aim of these chapters is to present the logic behind Frege's logicism and it contains the main aspects of Heck's interpretation. The second part of *Reading Frege's Grundgesetze* focuses on more formal aspects that make up part II of *Grundgesetze*. It contains six chapters and a chapter of appendices. While the first three chapters draw on previously published work, the remaining chapters contain mostly new material. Here, the aim is to present the mathematics behind Frege's Logicism and even though there are interesting philosophical discussions to be found, the main focus is a detailed discussion of Frege's proofs and more generally his proof strategy which as Heck argues is sometimes very telling. So this part functions more as a companion guide to Frege's formal achievements, which is, however, spiced up with many interesting philosophical insights and some intriguing conjectures.

After a brief preface which, in a Fregean manner, notes numerous reasons why the book was long in the making (the most notable one: "it had been made clear to me that further work on Frege and the philosophy of mathematics was not going to get me tenure" p.xii, fn3.) the book proper starts with a 24 page introduction which helpfully summarises the main aims and structure while also setting Frege's book into the wider context. In addition, chapter 1 motivates and presents some of the major aspects of Heck's interpretation of Frege's Logicism amongst which, for example, the claim that Frege knew

[1]Disclaimer: I guess I should say that I did not need much convincing having spent many years in a collaborative translation project of *Grundgesetze*.

of *Frege's Theorem*, i.e. that Hume's Principle—the neo-Fregean's favourite abstraction principle and rejected by Frege in *Grundlagen* as a suitable definition of the concept of number—embedded in (standard) second-order logic suffices to capture Peano arithmetic. By looking in detail at Frege's proofs in the second part of the book, Heck provides further detailed evidence for the claim that, in some sense, Frege uses value-ranges mainly for convenience.

Chapter 2 contains a very detailed discussion of Heck's semantic interpretation of Frege's logic—a theme he returns to on numerous occasions in subsequent chapters. For Heck, semantic notions play a crucial and constitutive role in Frege's conception of logic. Heck draws on Frege's criticism of Formalism which takes centre stage in vol. II of *Grundgesetze* to support his interpretation. A successful rejection of Formalism, Heck notes, incurs an obligation to justify the rules of inference by drawing on the "reference of the signs" (Frege, 2013, vol.II, p.156). As such, Heck convincingly argues that unless Frege is interpreted as entertaining some rather surprising double standards, we simply have to regard Frege's considerations in the passages between §5 to §25 as intended to provide a form of semantic justification for his laws and rules. But, as Heck urges, the kind of justification is not aimed at a full-blown sceptic, it is merely meant to provide some (presumably significant) epistemic ground to regard his basic laws (rules) as true (truth-preserving), and indeed as *logical*. In that context, Heck also makes the important (and to my mind correct) observation that Frege's discussion of Basic Law V in the Introduction to *Grundgesetze* is not to be understood as raising "doubts" about its truth, as most interpreters have done previously, rather Frege here merely acknowledges that a dispute about its *logical* status may arise.

Chapter 3-5 then focus in large parts on what must be the most difficult and from an interpretative perspective most challenging passages in Frege's *Grundgesetze*, §10 and §§29-32. In these sections, Frege appears to offer informal arguments that concern the semantic properties of his formal system. In particular, they purport to show that every well-formed expression refers. Of course, we know that Frege's arguments must ultimately fail but that's exactly where the interpretative challenge lies: one can't make the mistake too obvious, otherwise Frege should have really known better. In chapter 3, Heck discusses at length the structure of the argument of §29-31 and along the way discusses many other important issues. For example, he rejects a substitutional interpretation of Frege's quantifiers (p.56-57 and also later p.80), and in section 3.2 provides an extremely illuminating and much recommended discussion of Frege's use of Roman letters. The main feature of Heck's presentation is how he nicely manages to isolate different aspects of Frege's argument in §29-31. So, while, according to Heck, Frege's arguments have shown that he has in effect "produce[d] an informal, axiomatic theory of truth for the logical fragment of begriffschrift [i.e. Frege's formal language without the smooth breathing] and then [...] prove[d], again informally, that the theory in question is adequate, in roughly Tarski's sense" (p.71), Frege's argument to establish that the smooth breathing ("$\dot{\varepsilon}\Phi(\varepsilon)$") refers to value-ranges is in important (and ultimately fatal) ways different to his arguments for the other primitives. The so-called *Initial Stipulation* (offered in §3) fixes truth-conditions for identity statements involving the smooth breathing occurring on both sides. That is, however, it does not tell us whether an object not given *as* a value range is one. In effect, Frege's stipulation fails to explain "$\dot{\varepsilon}\Phi(\varepsilon)$" as names of objects and is thus, like Hume's Principle in *Grundlagen*, subject to (a version of) the Caesar Problem.

It is then the purpose of chapter 4 to discuss in detail §10 where Frege introduces further stipulations that add to the *Initial Stipulation* and explain why Frege thought he could legitimately disregard Caesar-like concerns. Like before, this chapter has much more to offer than a detailed and rich textual discussion of §10. It offers a succinct and clear discussion of the Caesar Problem which ties the discussion of §3, §9, and §10 of *Grundgesetze* to Frege's famous *Grundlagen* passages of §62ff. It provides a very intriguing and challenging discussion of Frege's own attitude towards Basic Law V, which again draws heavily on Heck's semantic interpretation of Frege's conception of logic. It is, or so Heck claims, only on the basis of being able to distinguish the *Initial Stipulation* from Basic Law V proper that Frege can ward off the accusation that he is putting forth a creative definition. Further, and controversially, Heck defends the view that Frege does not take his quantifiers to be unrestricted, as many interpreter have usually thought. Taking his lead from passages like: "so far we have only introduced the truth-values and value-ranges as objects" (Frege, 2013, vol.I, p.17) Heck interprets Frege as regarding the quantifiers of his formal language to be restricted and defends it against the charge that such a conception undermines Frege's attack against piecemeal definitions by attributing to Frege awareness of (something like) a distinction between domain of quantification and universe of discourse.

Chapter 5 then returns to §31 to provide an explanation for why, if Frege could not properly resolve the Caesar Problem, he thought he has done enough, as Heck puts it, to defang it. It is here then that Heck draws together the main insights of chapter 3 and 4 while discussing in detail Linnebo's alternative take on §§29-31. Heck's interpretation shows very nicely that Frege's argument that the smooth breathing refers is, ultimately, circular, while also showing that this circularity is in no way all that obvious. The argument depends ultimately on the assumption that no object is the value-range of more than one function, which however is what the proof is meant to show. Hence, it is Frege's failure to properly appreciate the pervasiveness of Caesar-like concerns which is ultimately responsible for the collapse of his argument.

What will be particularly helpful for readers of these crucial passages, is that Heck often sets Frege's thinking into a more familiar setting, so for example, Heck draws illuminating analogies to Tarski's work on numerous occasions and shows how Frege's motivating ideas behind his proof of referentiality are similar to the ideas behind Henkin's proof of completeness of the first-order functional calculus.

The second part of *Reading Frege's Grundgesetze* reads overall more like a detailed companion to Frege's formal proofs and definitions but again Heck manages to put the formal material into a wider context and shows for example how *Grundgesetze* draws on and corrects the informal proofs offered in *Grundlagen* (chapter 6). Readers interested in Frege's derivation of Hume's Principle and other "simple" truths of arithmetic will be immensely helped by Heck's detailed discussion and thoughtful presentation in that chapter. More generally, Heck really makes vivid just how much ground Frege covers in part II of *Grundgesetze*. To list just a few highlights: in chapter 7, Heck discusses Frege's proof that there are axioms of arithmetic (different in interesting ways to those of Dedekind) that characterise the natural numbers up to isomorphism. In chapter 8, we find a detailed discussion of Frege's theorems 328 and 348 that show how the natural numbers conceived as finite ordinals can be reconstructed from finite cardinals and we find a nice discussion of the closeness of Frege's and Zermelo's notion of finitude. Chapter 9 features Frege's proof that every subset of a countable set is countable, and Heck uses it to conjecture that Frege knew that Hume's Principle could not be a basis for a theory of real numbers. Frege's discussion of addition takes centre stage in chapter 10. Lastly, in chapter 11, Heck draws on the fact that Frege had read Dedekind's *Was sind und was sollen die Zahlen?* just before the publication of *Grundgesetze* to explain that a rather odd collection of results about the infinite is best understood in the context of an intellectual engagement with Dedekind. Interestingly, Heck shows how Frege's notion is very different to Dedekind's and argues that on that basis Frege became aware of the axiom of choice. In a way, it is simply astonishing just how much Heck manages to cover in the second part of his book and how many fascinating connections to either Frege's own contemporaries or more recent thinking are revealed.

Throughout the first and second part, Heck translates Frege's formalism into modern notation. But as Heck himself acknowledges, Frege's formalism isn't difficult to read, it is difficult to learn to read. Given that he does not use Frege's original symbols and given that following Frege too closely would lead to confusions in a more modern formalism (especially with respect to bound/free variables), I did initially find that translating back and forth between Heck's and Frege's formalism is not straightforward. However, after a while one gets used to it and Heck's translations offer a valuable guide to working with Frege's original formalism and so help the reader to learn to read the original.

In summary, Heck succeeds in drawing together many different philosophical and formal aspects of Frege's impressive intellectual achievements and thus pays a fitting tribute to a thinker who really was "before his time". I highly recommend this book—it is, simply put, fantastic and full of deep insights and challenging ideas. *Reading Frege's Grundgesetze* is, on the one hand, an accessible guide to Frege's magnum opus that will offer much needed help to first-time readers of *Grundgesetze*. On the other, even hardened Frege scholars will benefit and will, no doubt, be challenged by Heck's detailed and thought-provoking interpretation of Frege's mature philosophy of logic and mathematics.

Now, there is, however, one glaring "omission" in Heck's discussion: there is no detailed treatment of part III which covers much of volume II of *Grundgesetze*. Although Heck draws selectively on some of the criticisms that Frege offers in the informal parts of part III, Heck raises some doubts about the potential philosophical interest of Frege's formal development of the real numbers, of which, admittedly, there is little to be found in volume II. Maybe history has something to teach us here: since Heck's book we know how off the mark scholars were in thinking that there is little of interest in *Grundgesetze* simpliciter, and so one may hope that maybe some day someone will manage to convince Heck that there is much that is of interest in Frege's sketchy treatment of the real numbers. For what it is worth, I have no doubt that Heck, as he himself admits, "would love to be proven wrong".

References

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