Structures, homomorphisms, and the needs of model theory

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Abstract: When we look closely at textbooks on model theory, we find that there are three different accounts of what a model or structure is. One of these is highly language dependent, so that the same structure cannot be the interpretation of two different languages or signatures. The other two definitions do not fall foul of that dependence but *all* textbooks tie the notion of homomorphism so closely to language (signature) that only structures interpreting the same language (signature) are isomorphic. Although this follows the practice in universal algebra, it is highly unnatural. The aim here is to present a notion of homomorphism better consonant with intuition and with what the less cautious authors of textbooks say when they speak informally.

Keywords: Model theory, structures, homomorphisms

1 Introduction

There is an informal notion of a *structure* which one finds in many texts on model theory, a notion borrowed from algebra. There is also, in many, a formal notion supposedly motivated by the informal one. Depending on which text one reads, these two notions may not match up: it depends on the way in which the formal notion involves a language.[1] To see what I'm getting at, begin by considering Wilfrid Hodges' characterization of contemporary practice in model theory:

> Model theorists are forever talking about symbols, names, and labels. A group theorist will happily write the same abelian group multiplicatively or additively, whichever is more convenient for the matter in hand. Not so the model theorist: for him

[1] In some texts, such as the classic *Model Theory* (Chang & Keisler, 1990), 'structure' is used only informally and does not appear in the index; on the other hand, 'model' is used both formally and informally.

> or her the group with '·' is one structure and the group with '+' is a different structure. Change the name and you change the structure. (Hodges, 1993, p. 1; Hodges, 1997, p. 1)[2]

If we read this literally, it has an odd consequence, for many of the texts in model theory published in English over the past fifteen years or so started life in other languages: (Manzano, 1999), Spanish; (Poizat, 2000), French; (Rothmaler, 2000), (Prestel & Delzell, 2011), German; (Tent & Ziegler, 2012), has its origin in lecture notes in German; (Marcja & Toffalori, 2003) is not a literal translation but has some overlap with (Marcja & Toffalori, 1998). So many languages—so many different structures? However odd that may seem, on closer inspection of what these and other authors say, we find that there are in practice *three different accounts* of what a model-theoretic structure (or model) is.

Hodges continues,

> This must look like pedantry. [...] Nevertheless there are several good reasons why model theorists take the view that they do. [...]
>
> In the first place, we often want to compare two structures and to study the homomorphisms from one to the other. What is a homomorphism? [...A] homomorphism from structure A to structure B is a map which carries each operation of A to *the operation with the same name in* B. (Hodges, 1993, p. 1; Hodges, 1997, p. 1; emphasis in the originals)

It is an immediate and surely unwanted—and, I think, largely unrecognised—consequence of what Hodges tells us here that where the algebraist sees only one group Hodges' model theorist must see two *non-isomorphic* groups. But worse is to come. Even those authors who can readily agree with the algebraist that there is only one structure give the language (or the signature) used in the specification of a structure such a role in the definition of *homomorphism* that structures specified using different languages/signatures simply *cannot* be isomorphic! We have, then, an unhappy situation: either we have (i) a notion of structure compatible with the notion of homomorphism but which renders structure so parochial that the same model theorist speaking in different languages cannot, by her own lights, be speaking of the same structures, or we have (ii) a notion of structure consonant with common sense and a notion of homomorphism so parochial that

[2] (Hodges, 1997) takes a lot of material unchanged from (Hodges, 1993).

what should count as the *same* model specified in different languages has to be seen as a family of necessarily non-isomorphic models.

The definition of homomorphism that we find in (Hodges, 1993, 1997), (Poizat, 2000), (Rothmaler, 2000), (Marker, 2002), (Marcja & Toffalori, 2003), (Prestel & Delzell, 2011), (Tent & Ziegler, 2012) takes homomorphisms to be defined *only* between structures with a common signature. Despite this, Hodges says that it 'is meant to take in, with one grand sweep of the arm, virtually all the things that are called "homomorphism" in any branch of algebra'! And truth to tell, textbooks in universal algebra conform. But intuitively the notions of homomorphism and isomorphism that these authors arrive at fall *far* short of the mark. Indeed, going by the letter of the definitions of structure and homorphism they offer, we can have languages interpreted in the same domain with the same distinguished individuals, functions and relations but the resulting structures are *not* isomorphic exactly because no allowance for difference of language is made in the definition of homomorphism. Authors who do not tie structures to languages—*e.g.*, Bell and Slomson (1969), Sacks (1972), Manzano (1999)—do no better, for they too, as we'll see, tie homomorphisms to signatures (similarity types).

My aims are twofold: to tease out the various notions of structure in the textbooks and, rather more importantly, to arrive at a notion of homomorphism compatible with the less parochial reading of structure. For help in this, I turn to Joseph Goguen and Rod Burstall's theory of institutions (Goguen & Burstall, 1992); here we find a stepping-stone towards a definition of homomorphism between structures that better matches what the less cautious of our authors say when they speak informally. (Though the morals drawn have wider application, I shall, as is common in textbook presentations of model theory, confine attention to the model theory of first-order languages.)

2 What is model theory?

Let us begin by taking a look at what some texts on model theory say model theory is. In J. L. Bell and A. B. Slomson's *Models and Ultraproducts* (1969) we find

> Model theory ... can be described briefly as the study of the relationship between formal languages and abstract structures.
> (Bell & Slomson, 1969, p. 1)

David Marker, *Model Theory: An Introduction* (2002):

> Model theory is a branch of mathematical logic where we study mathematical structures by considering the first-order sentences true in those structures and the sets definable by first-order formulas. (Marker, 2002, p. 1)

Annalisa Marcja and Carlo Toffalori, *A Guide to Classical and Modern Model Theory* (2003):

> Model Theory is—or, more precisely, was at its beginning—the study of the relationship between mathematical formulas and structures satisfying or rejecting them. (Marcja & Toffalori, 2003, p. 1)

I fully expect that you find nothing controversial in these claims. What I want to point to here is that they are naturally read as agreeing with what Bell and Slomson say explicitly:

> In ch. 3 we described the language L of predicate calculus with equality and then we looked at interpretations, i.e. realizations, of this language. In this chapter and for most of the rest of this book we are going to look at things the other way round. We regard relational structures as the objects of primary interest and we introduce a first order predicate language in order to be able to say things about them. This is the most natural viewpoint in mathematics; *it is the structures that present themselves first.* (Bell & Slomson, 1969, p. 72, my emphasis)

As Gerald Sacks says pithily, 'The "objects" of model theory are the structures' (Sacks, 1972, p. 10). On the face of it, this is all very much at odds with what we saw Hodges saying.

Now, I certainly do not claim that there is conceptual confusion, far less error, in the actual day-to-day practice of model theory. But there are issues concerning basic notions, issues which become especially pressing if one thinks of model theory as providing a paradigm for semantics. That the identity of what we talk about be in part determined by the language in which we talk, for example, would render translation impossible, as it might seem to be, on Hodges' characterization, in the case of our model theory textbooks, for we cannot refer in one language to what we speak of in another.

3 Structures in the intuitive sense

Let's take a quick look at what some of the textbooks say about structures. Chang and Keisler begin *Model Theory* by saying

> Let us now take a short introductory tour of model theory. We begin with the models which are structures of the kind that arise in mathematics. For example, the cyclic group of order 5, the field of rational numbers, and the partially-ordered structure consisting of all sets of integers ordered by inclusion, are models of the kind we consider. (Chang & Keisler, 1990, p. 1)

In similar fashion, in his chapter on first-order logic in the model theory section of the *Handbook of Mathematical Logic*, Jon Barwise gives us groups, a group being a triple $\langle G, +, 0\rangle$ satisfying certain conditions, namely associativity of the operation $+$, the existence of an identity, 0, for $+$, and the existence of inverses with respect to $+$ and the identity, and the ordered field $\mathfrak{R} = \langle \mathbb{R}, +, \cdot, <, 0, 1\rangle$ of real numbers among other examples (Barwise, 1977, pp. 7-8, 11).

Philipp Rothmaler (*Introduction to Model Theory*, 2000) offers

> By specifying certain neutral elements and operations, we may view the set $\mathbb{Z}$ of integers as an additive group, as a multiplicative semigroup, or as a ring. [...] We could also add the inverse operation $-$ or the ordering relation $<$. (Rothmaler, 2000, p. 3)

Marker says

> Intuitively a structure is a set that we wish to study equipped with a collection of distinguished functions, relations, and elements. We then choose a language where we an talk about the distinguished functions, relations, and elements and nothing more. For example, when we study the ordered field of real numbers with the exponential function, we study the structure $(\mathbb{R}, +, \cdot, \exp, <, 0, 1)$ where the underlying set is the set of real numbers, and we distinguish the binary functions addition and multiplication, the unary function $x \mapsto e^x$, the binary order relation, and the numbers 0 and 1. To describe this structure we could use a language where we have symbols for +, $\cdot$, exp, $<$, 0, 1 ....

> For another example, we might consider the structure $(\mathbb{N}, +, 0, 1)$ of the natural numbers with addition and distinguished elements 0 and 1. The natural language for studying this structure is the language where we have a binary function symbol for addition and constant symbols for 0 and 1. (Marker, 2002, p. 7)

Marcja and Toffalori say,

> Structures are an algebraic notion. [...] What is a structure? Basically, it is a non-empty set A, with a collection of distinguished elements, operations, and relations. For instance, the set **Z** of integers with the usual operations of addition + and multiplication · is a structure, as well as the same set **Z** with the order relation $\leq$. Note that in these cases the underlying set is the same (the integers), but, of course, the structure changes: in the former case we have the ring of integers, in the latter the integers as an ordered set. (Marcja & Toffalori, 2003, p. 1)

But some authors do follow Hodges' line. Like Hodges, Bruno Poizat (2000), Alexander Prestel and Charles Delzell (2011), and Katrin Tent and Martin Ziegler (2012) are careful to spell out the language first when presenting a structure—in flat out contradiction, it must be said, of what Bell and Slomson called the most natural viewpoint in mathematics.

Hodges, however, rather gives the game away with one of his examples—either that or he credits naming with magical power! He says:

> **Linear orderings** Suppose $\leq$ linearly orders a set X. Then we can make $\langle X, \leq \rangle$ into a structure A as follows. The domain of A is the set X. There is one binary relation symbol R, and its interpretation R^A is the ordering $\leq$. (Hodges, 1993, p. 3; Hodges, 1997, p. 3)

For Hodges, going by what he says informally, the ordered pair $\langle X, \leq \rangle$ is *not* a structure, not, at least, for the model theorist—it might be for the algebraist. *We* make it into one by taking a two-place relation symbol and assigning $\leq$ as its interpretation. This gives rise to a bizarre profusion of entities. Suppose you pick one relation symbol and I pick another—we now have two structures where previously we had none. How many structures built over $\langle X, \leq \rangle$ are there? Hodges is, quite explicitly, not at all fussed about what counts as a language.

> Aha—says the group theorist—I see you aren't really talking about *written* symbols at all. For the purposes you have described, you only need to have formal labels for some parts of your structures. It should be quite irrelevant what kinds of thing your labels are; you might even want to have uncountably many of them.
>
> Quite right. In fact we shall follow the lead of A. I. Mal'tsev (1936) and put no restrictions at all on what can serve as a name. For example, any ordinal can be a name, and any mathematical object can serve as a name of itself. (Hodges, 1993, p. 2; Hodges, 1997, p. 2)

Now, if that is right, $\langle X, \leq \rangle$ *is* a structure with $\leq$ autonymously naming itself and we are not needed to make it into one (which, to be fair, I don't really suppose Hodges thinks we are) and, assuming we aren't really needed to associate names and what they name, there are at least as many structures built over $\langle X, \leq \rangle$ as there are items in your favourite set-theoretic hierarchy. And—the main point, to be developed below—they are all isomorphic in a way that model-theorists do *not* capture with their definitions of isomorphism between structures!!!

4 Structures in the formal sense

Minor variations aside, there are *three* styles of definition for models or structures in textbooks on model theory.

Definitions of the first kind: labelling (1) Let L comprise the non-logical vocabulary of a first-order language. A model/interpretation/relational structure $\mathfrak{A}$ for L is an ordered pair $\langle A, \mathscr{I} \rangle$, where

- ◦ A is a non-empty set;
- ◦ $\mathscr{I}$ assigns elements of A to the constants in L, n-ary operations on A to the n-place function-symbols of L, and n-place relations over A to the n-place relation-symbols of L.

We call $\mathscr{I}$ an *interpretation function*. We call L the *signature* of $\mathfrak{A}$.

Definitions of the second kind: labelling (2) Let L and $\mathscr{I}$ be as above. A model/interpretation/relational structure $\mathfrak{A}$ for L is an ordered pair $\langle A, \mathscr{I}[L]\rangle$, where $\mathscr{I}[L]$ is the image of L under $\mathscr{I}$.

As $\mathscr{I}$ is a function from L into $A \cup \bigcup_{n\in\mathbb{N}^+} A^{A^n} \cup \bigcup_{n\in\mathbb{N}^+} \mathscr{P}(A^n)$, $\mathscr{I}[L]$ is some non-empty subset of $A \cup \bigcup_{n\in\mathbb{N}^+} A^{A^n} \cup \bigcup_{n\in\mathbb{N}^+} \mathscr{P}(A^n)$.

The difference—a difference that has, of course, no real significance for the practice of model theory but is nonetheless present right in the very characterization of model theory's objects of study!—is this: for some authors a model comprises a domain A and a *function* from a language/signature into the set $A \cup \bigcup_{n\in\mathbb{N}^+} A^{A^n} \cup \bigcup_{n\in\mathbb{N}^+} \mathscr{P}(A^n)$; for others it is the domain together with the *image* of the language/signature under such a function. Often enough the distinction is collapsed notationally: while a model is said to comprise a domain and function, it is written out as a domain together with the image of the language/signature under that function. For example, Annalisa Marcja and Carlo Toffalori tell us

> A structure $\mathcal{A}$ for L is a pair consisting of a non empty set A, called the **universe** of $\mathcal{A}$, and a function mapping
>
> (i) every constant c of L into an element $c^{\mathcal{A}}$ of A,
>
> and, for any positive integer n,
>
> (ii) every n-ary function symbol f of L into an n-ary operation $f^{\mathcal{A}}$ of A (hence a function from A^n into A),
>
> (iii) every n-ary relation symbol R of L into an n-ary relation $R^{\mathcal{A}}$ of A (hence a subset of A^n).
>
> The structure $\mathcal{A}$ is usually denoted as follows
>
> $$\mathcal{A} = (A, (c^{\mathcal{A}})_{c\in L}, (f^{\mathcal{A}})_{f\in L}, (R^{\mathcal{A}})_{R\in L}).$$
>
> (Marcja & Toffalori, 2003, p. 2)[3]

Chang and Keisler, more careful than most, distinguish between a model and its "displayed form" (Chang & Keisler, 1990, p. 20).

[3] $(c^{\mathcal{A}})_{c\in L}$, $(f^{\mathcal{A}})_{f\in L}$, and $(R^{\mathcal{A}})_{R\in L}$ are indexed families (or indexed sets). On some accounts, indexed families are functions (with the indexing set as domain and the set of indexed elements as the range). But Sacks is clear that, *e.g.*, $\{c_k^{\mathcal{A}} | k \in K\}$ in his notation is a *subset* of the domain A (Sacks, 1972, p. 8) and Hodges is likewise so (Hodges, 1993, p. 2, Hodges, 1997, p. 2). Moreover, what would be the point of replacing one function with three?

We should note that there are two ways to think of an indexed family such as $(c^{\mathcal{A}})_{c\in L}$. One

Definitions of the third kind: indexing A (similarity) type or signature is a quintuple $\langle I, J, K, \mu, \nu \rangle$ such that $\mu : J \to \mathbb{N}$ and $\nu : K \to \mathbb{N}$.

A model/relational structure $\mathfrak{A}$ of signature/type $\langle I, J, K, \mu, \nu \rangle$ consists of

- a non-empty set A;
- for each $i \in I$, an element $c_i^{\mathfrak{A}}$ of A;
- for each $j \in J$, a $\mu(j)$-ary operation $f_j^{\mathfrak{A}}$ defined on A;
- for each $k \in K$, a $\nu(k)$-place relation $R_k^{\mathfrak{A}}$ defined over A.

The sets I, J and K index the distinguished individuals, functions and relations in the model, respectively. We can again think of there being an interpretation function, or, better, three interpretation functions, from the indexing sets into A, $\bigcup_{n \in \mathbb{N}^+} A^{A^n}$, and $\bigcup_{n \in \mathbb{N}^+} \mathscr{P}(A^n)$, respectively.

Gerald Sacks (*Saturated Model Theory*) takes the signature/type to be the quintuple; María Manzano (*Model Theory*) takes it to be just $\langle \mu, \nu \rangle$ — she takes individuals to be functions of zero arguments but notes that we could distinguish them from functions.

We have a 2×2 classification of what structures/models are said to be: they divide on whether the signature σ is the non-logical vocabulary L of a language $\mathscr{L}$ or comprises arbitrary indexing sets and on whether the second component of a model is an interpretation function $\mathscr{I}$ or the image of the signature under that function $\mathscr{I}[\sigma]$. In the case of indexing, the model is given as the images of the two (Manzano) or three (Sacks) indexing functions; what's important is that it's the images, not the functions themselves, that are constituents of the model. Table 1 encapsulates where various authors stand.

Here σ is the type or signature. When $\mathfrak{A} = \langle A, \mathscr{I} \rangle$, we have Hodges' "different language, different structure". When $\mathfrak{A} = \langle A, \mathscr{I}[\sigma] \rangle$, the same structure can be specified using different signatures.[4]

way respects the multiplicity of multiple occurrences, with different indices, of the same item. This way, we would obtain a *multiset*. The other is to ignore repetitions, thus taking $(c^{\mathcal{A}})_{c \in L}$ and its ilk to be sets (as I have in speaking of the image of the signature under the interpretation function). None of our authors explicitly endorses the multiset reading; some are explicit in insisting on sets. On multisets, see, *e.g.*, (Hickman, 1980; Blizard, 1989, 1991; Singh, 1994; Singh, Ibrahim, Yohanna, & Singh, 2007).

[4]Textbooks on universal algebra mostly fall into the lower right quadrant (*e.g.* Grätzer, 1979, pp. 8 & 223-4; Cohn, 1981, pp. 48 & 188-9; Denecke & Wismath, 2002, p. 4; Bergman,

Table 1: What is a structure?

	$\sigma = L$ (labelling)	$\sigma \neq L$ (indexing)
$\mathfrak{A} = \langle A, \mathscr{I} \rangle$	Chang and Keisler 1990 Barwise 1977 Poizat 2000 Marcja and Toffalori 2003	
$\mathfrak{A} = \langle A, \mathscr{I}[\sigma] \rangle$	Hodges 1993, 1997 Rothmaler 2000 Marker 2002 Prestel and Delzell 2011 Tent and Ziegler 2012	Bell and Slomson 1969 Sacks 1972 Manzano 1999

The placing of Hodges here may strike the reader as odd given our starting point. What Hodges says is

> A **structure** A is an object with the following four ingredients.
>
> (1.1) A set called the **domain** of A ...
>
> (1.2) A set of elements of A called **constant elements**, each of which is named by one or more **constants**. If c is a constant, we write c^A for the constant element named by c.
>
> (1.3) For each positive integer n, a set of of n-ary relations ... each of which is named by one or more n-ary **relation symbols**. If R is a relation symbol, we write R^A for the relation named by R.
>
> (1.4) For each positive integer n, a set of of n-ary operations ... each of which is named by one or more n-ary **function symbols**. If F is a function symbol, we write F^A for the function named by F.
>
> (Hodges, 1993, p. 2, Hodges, 1997, p. 2)

2012, pp. 3-4). A couple speak of symbols and look more like they belong in the bottom left (*e.g.* Wechler, 1992, pp. 5-6; Plotkin, 1994, pp. 36-7). I owe to Göran Sundholm the suggestion to investigate what textbooks in universal algebra say.

I leave it to the reader to judge the accuracy of my placing.[5]

5 Homomorphisms

Every one of our authors ties homomorphisms to signatures (types).

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of the same signature L with domains A and B respectively, determined by two interpretation functions, $\mathscr{I}$ and $\mathscr{J}$, respectively (both with domain L).

A function $h : A \to B$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ iff

- $h(\mathscr{I}(c)) = \mathscr{J}(c)$, for each constant c in L;
- for each $\langle a_1, a_2, \ldots, a_n\rangle$ in A, $h(\mathscr{I}(f)(a_1, a_2, \ldots, a_n)) = \mathscr{J}(f)(h(a_1), h(a_2), \ldots, h(a_n))$, for each n-ary function-symbol f in L;
- for each $\langle a_1, a_2, \ldots, a_n\rangle$ in A, if $\mathscr{I}(R)(a_1, a_2, \ldots, a_n)$ then $\mathscr{J}(R)(h(a_1), h(a_2), \ldots, h(a_n))$, for each n-place relation-symbol R in L.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of the same signature/type $\langle I, J, K, \mu, \nu\rangle$ with domains A and B respectively.

A function $h : A \to B$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ iff

- $h(c_i^{\mathfrak{A}}) = c_i^{\mathfrak{B}}$, for each i in I;

[5] Hodges defines signatures thus:

> The **signature** of a structure A is specified by giving
>
> > the set of constants of A, and, for each $n > 0$, the set of n-ary relation symbols and the set of n-ary function symbols of A.
>
> We shall assume that the signature of a structure can be read off uniquely from the structure.
>
> The symbol L will be used to stand for signatures. Later it will also stand for languages – think of the signature of A as a rudimentary language for talking about A. If A has signature L, we say A is an L-structure. (Hodges, 1993, pp. 4-5; Hodges, 1997, pp. 4-5)

Given that Hodges allows each item—object, n-ary relation, n-ary operation—in a structure to be named by more than one element of the signature, one might wonder how we read the signature off the structure. I guess the thought is that in *presenting* a structure we cannot but associate the items comprising the structure with elements of a signature/language.

- for each $\langle a_1, a_2, \ldots, a_{\mu(j)} \rangle$ in A,
  $h(f_j^{\mathfrak{A}}(a_1, a_2, \ldots, a_{\mu(j)})) = f_j^{\mathfrak{B}}(h(a_1), h(a_2), \ldots, h(a_{\mu(j)}))$, for each j in J;

- for each $\langle a_1, a_2, \ldots, a_{\nu(k)} \rangle$ in A, if $R_k^{\mathfrak{A}}(a_1, a_2, \ldots, a_{\nu(k)})$ then $R_k^{\mathfrak{B}}(h(a_1), h(a_2), \ldots, h(a_{\nu(k)}))$, for each k in K.[6]

So even when $\langle A, \mathscr{I}[\sigma] \rangle = \langle A, \mathscr{J}[\sigma'] \rangle$, the identity function on A cannot count as a homomorphism unless $\sigma = \sigma'$.

In the intuitive sense, a structure is a set A, its domain, together with some subset of $A \cup \bigcup_{n \in \mathbb{N}^+} A^{A^n} \cup \bigcup_{n \in \mathbb{N}^+} \mathscr{P}(A^n)$ comprising the distinguished individuals, functions, and relations. With that in mind, there are two ways we might go from here. One is to broaden the model-theorist's notion of homomorphism to allow homomorphisms between structures *in the formal sense* with different signatures. The other is to define a notion of homomorphism appropriate to structures *in the intuitive sense*. Although textbooks in model theory ignore both, it's fairly clear how to go about getting both.

Firstly, we borrow from Joseph Goguen and Rod Burstall's theory of *institutions* the idea of a *signature morphism* (Goguen & Burstall, 1992). A signature morphism is a function between signatures mapping constants to constants, n-ary function-symbols to n-ary function-symbols, and n-place relation-symbols to n-place relation-symbols.[7] What we're after is a conception of a homomorphism from a model with signature L to a model

[6] In their definition of *homomorphism*, Bell and Slomson (1969, p. 73) strengthen this to an 'if, and only if', thus defining what Manzano (1999, p. 24) calls a 'strong homomorphism'. *Cf.* the definitions of *monomorphism* in (Sacks, 1972, p. 9), where additionally injectivity of h is required.

[7] I owe thanks to Roy Dyckhoff for drawing the literature on institutions to my attention. Although signature morphisms are the essential stepping-stone to where we want to get to here, Goguen and Burstall's aims are different; they do not define a notion of homomorphism between structures with different signatures.

Goguen and Burstall are interested in the categorial structure you get when you put signature morphisms together with homomorphisms of structures (in the formal sense) for fixed signatures. In particular, if i is a signature morphism from L to L' then there's an induced mapping from the category of models with signature L' into the category of models with signature L—notice the reversal of the order. (The morphisms in the categories are the homomorphisms in the formal sense between structures with the same signature—L' and L, respectively.)

Given a signature morphism $i : L \to L'$ and a model with domain A and signature L', determined by interpretation function $\mathscr{J}$, we define the associated model with domain A and signature L by defining the interpretation function $\mathscr{I}$ as follows:

- $\mathscr{I}(c) = \mathscr{J}(i(c))$, for all constants c in L;

with signature L'. We can get that by combining our previous definition of a homomorphism for models of the same signature with a signature morphism.What we get is the following:

$h : A \to B$ is a homomorphism from the model $\mathfrak{A}$, with domain A, determined by signature L and interpretation function $\mathscr{I}$, to the model $\mathfrak{B}$, with domain B, determined by signature L' and interpretation function $\mathscr{J}$ if there's a signature morphism i from L to L' and

- $h(\mathscr{I}(c)) = \mathscr{J}(i(c))$, for each constant c in L;
- for each $\langle a_1, a_2, \ldots, a_n \rangle$ in A, $h(\mathscr{I}(f)(a_1, a_2, \ldots, a_n)) = \mathscr{J}(i(f))(h(a_1), h(a_2), \ldots, h(a_n))$, for each n-ary function-symbol f in L;
- for each $\langle a_1, a_2, \ldots, a_n \rangle$ in A, if $\mathscr{I}(R)(a_1, a_2, \ldots, a_n)$ then $\mathscr{J}(i(R))(h(a_1), h(a_2), \ldots, h(a_n))$, for each n-place relation-symbol R in L.[8]

The guiding thought is that we can have a homomorphism not only between structures that *are* labelled by the members of the same signature but between structures that *can* be labelled by the members of the same signature. We can limit attention to the case when the signature morphism is surjective; indeed, we had better if we are not to lose sight of reducts and expansions. (Obviously, we can draw up an analogous definition for the case of structures defined by types. So too in what follows immediately below.)

A structure in the intuitive sense *serves as an interpretation* of the signature L if there is an interpretation function $\mathscr{I}$ from L *onto* the set of distinguished items of the structure. The structure *serves as a duplication-free*

- $\mathscr{I}(f) = \mathscr{J}(i(f))$, for all n-ary function-symbols f in L;
- $\mathscr{I}(R) = \mathscr{J}(i(R))$, for all n-place relation-symbols R in L.

It now turns out that if there's a homomorphism (in the already defined sense) from the model with domain A determined by interpretation-function $\mathscr{J}$ to the model with domain B determined by interpretation-function $\mathscr{J}'$, both with signature L', then the same function from A to B is a homomorphism from the associated model with domain A and signature L induced by i to the associated model with domain B and signature L induced by i—and, if you think about it, that's obvious.

[8] For a strong homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ we require that the last clause be tightened to an 'if and only if' (Manzano, 1999, p. 24; Rothmaler, 2000, p. 5); for an embedding (monomorphism) of $\mathfrak{A}$ into $\mathfrak{B}$ we require further that h be injective. For an isomorphism we require, additionally, that it be surjective.

interpretation if $\mathscr{I}$ is a bijection. We can then say: there is a homomorphism from the structure with domain A to the structure with domain B if (*i*) they serve as duplication-free interpretations of the same signature L (under the interpretation functions $\mathscr{I}$ and $\mathscr{J}$, respectively) and (*ii*) there is a homomorphism from the structure in the formal sense with domain A, determined by L and $\mathscr{I}$, into the structure in the formal sense with domain B, determined by L and $\mathscr{J}$.

(Given a domain A, let S be some subset of $A \cup \bigcup_{n\in\mathbb{N}^+} A^{A^n} \cup \bigcup_{n\in\mathbb{N}^+} \mathscr{P}(A^n)$. If we follow the lead Hodges ascribes to Anatoliĭ Ivanovich Mal'tsev, the existence of a language for which $\langle A, S\rangle$ serves as a duplication-free interpretation is trivial: we take S itself to be the language, its elements naming themselves autonymously.)

6 The needs of model theory

A structure comprises a domain and a set of distinguished items (individuals, operations, relations). In order to distinguish those distinguished items, we need to use labelling or indexing (or something like that). So we naturally associate labels or indices with the distinguished items in a structure. What we shouldn't do—and, as we've seen, the majority of model-theory texts do not—is to take the labelling/indexing to be a feature of the structure. (Chang & Keisler, 1990, Barwise, 1977, Poizat, 2000, Marcja & Toffalori, 2003 are the exceptions.)

When we consider homomorphisms between structures, we need to indicate which individual/operation/relation in one structure matches up with which individual/operation/relation in the other. Using a common labelling/indexing is an easy way to spell that out. So the way model-theorists proceed is entirely natural. The problem then is this: if we leave matters this way, this ties our notion of homomorphism to a shared labelling and it is this that is at odds with the "pre-theoretical", ordinary, working mathematician's notion of a homomorphism between structures. As we saw, what should be one structure turns into a family of non-isomorphic structures when presented under different labellings/indexings.

Despite the central role languages play in model theory, model-theorists are not interested in particular languages (not in the way field linguists are); they are interested in classes of languages, languages in which certain kinds of thing can be said, distinctions drawn, and the like, and the relations of (classes of) languages to their interpretations. Now, any structure (in the intuitive sense) that serves as an interpretation of a signature L, serves equally as an interpretation of any other signature in which the various grammatical categories—constants, n-place function-symbols, n-place relation-symbols—have the same cardinalities. At least some of the time, this is what model-theorists have in mind. But model-theorists cannot make do with classes of languages distinguished solely by the cardinalities of their grammatical categories. Sometimes they start with models of one language then consider interpretations of expansions of that language. Expanding a language needn't change the cardinalities of its grammatical categories. Nevertheless, a distinction needs to be made between the original language and its expansion(s). And an interpretation of an expansion is exactly that: it is not usefully to be thought of as an interpretation of the original language.

The model-theorist cannot—or, at the very least, cannot without a good deal of unedifying, pointlessly pedantic, faffing about—reduce the identification of languages to the cardinalities of the categories of their non-logical vocabulary. Such an identification *is* all that's needed in determining whether a structure can serve as an interpretation of a language and for spelling out a sensible notion of homomorphism. But in practice it won't do for all model theory's purposes. Consequently, the model theorist ends up speaking as though she does have particular languages in mind (even though she really doesn't).

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